# Exact Analytical Solution of the Constrained Statistical Multifragmentation Model

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A rigorous theory of critical phenomena in finite systems was not built up to now. However, the experimental studies of phase transitions (PTs) in some systems demand the formulation of such a theory. In particular, the investigations of the nuclear liquid-gas PT require the development of theoretical approaches which would allow us to study the critical phenomena without going into the thermodynamic limit $V \to \infty$ ($V$ is the volume of the system) because such a limit does not exist due the long range Coulomb interaction. Therefore, there is a great need in the theoretical approaches which may shed light on the "internal mechanism" of how the PTs happen in finite systems. Traditionally, for infinite systems this role is played by the exactly solvable models with PT, since they provide us with the first principle information on PT which cannot be obtained either within the widely used mean-filed approach or numerically.

Recently a great deal of progress was achieved in our understanding of finite volume PTs, when an exact analytical solution of a simplified version of the statistical multifragmentation model (SMM) [1, 2] was extended to finite systems [3]. This result is obtained for a more general formulation of SMM, the constrained SMM (CSMM), in which the size of the largest nuclear fragment $bK(V)$ ($b$ is nucleon eigen volume, $K(V)$ is the number of nucleons) can be an arbitrary growing function of $V$ which does not exceed $V$, i.e. $bK(V) \leq V$.

An exact solution of CSMM was found with the help of a novel powerful mathematical method, the Laplace-Fourier transform, developed in [3]. The method is based on the following identity

$$G(V) = \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi}\, e^{i\eta(V-\xi)}\, G(\xi)\,, \qquad (1)$$

which allows one to transform an arbitrary volume dependence of function $G(V)$ into an exponential one. The latter can be easily managed by the traditional Laplace transform method in $V$ variable to the isobaric ensemble. As proven in [3], the grand canonical partition (GCP) of the CSMM can be *identically* rewritten as the sum over simple poles $\lambda_n$ ($n = 0, 1, 2, \ldots$) of the isobaric partition:

$$\mathcal{Z}(V,T,\mu) \;= \sum_{\{\lambda_n\}} e^{\lambda_n V} \left[1 - \tfrac{\partial \mathcal{F}(V,\lambda_n)}{\partial \lambda_n}\right]^{-1}. \qquad (2)$$

The simple poles $\lambda_n$ are defined by the following equation

$$\lambda_n \;=\; \mathcal{F}(V,\lambda_n) \equiv \sum_{k=1}^{K(V)} \phi_k(T)\, e^{\frac{(\mu - \lambda_n bT)k}{T}}. \qquad (3)$$

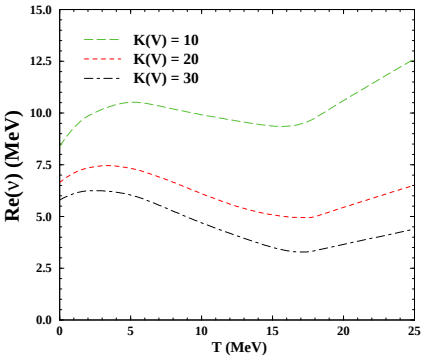


FIG. 1: Each curve separates the $T - Re(\nu_n)$ region of one real root of Eq. (3) (below the curve), three complex roots (at the curve) and five and more roots (above the curve) for three values of $K(V)$.

Here $\mu$ is chemical potential, $\phi_k(T) = g_k \left[\frac{mT}{2\pi}\right]^{\frac{3}{2}} e^{-\frac{f_k}{T}}$ is the one-particle distribution function of the fragment of $k$-nucleons, parametrized by its free energy $f_k$, the spin-isospin degeneracy $g_k$ and temperature $T$.

A detailed analysis of singularities $\lambda_n$ allows one to introduce the finite volume analogs of phases and make a direct link between them and the phases of infinite system. In thermodynamic limit $V \to \infty$ there are only two singularities which define the pressure of the gas $T\lambda_0(T,\mu)$ and the pressure of the liquid $T\lambda_L(T,\mu)$. Their effective chemical potentials are $\nu_0 \leq 0$ and $\nu_L = 0$, respectively. The rightmost singularity defines the stable phase. The phase transition occurs, when both singularities coincide, i.e. $T\lambda_0(T,\mu) = T\lambda_L(T,\mu)$, which can happen at $\nu_0 = \nu_L = 0$ only. The former equality is nothing else but the famous Gibbs criterion for PT.

For finite volumes and finite values of $\mu$ and $T$ the number of simple poles with $Re(\lambda_n) \geq 0$ is finite. The real pole $\lambda_0(T,\mu)$, like for an infinite volume, corresponds to a gas state, but now its effective chemical potential may become positive: $\nu_0(T) > 0$. The complex poles $\lambda_{n>0}$ generate complex values of the effective chemical potentials $\nu_{n>0}$. The analysis shows that for finite volumes the coexistence line $\nu_0 = \nu_L = 0$ of infinite system is transformed into a region of mixed phase for $\nu_0(T) \geq Re(\nu_{n>0}(T)) < \infty$ (see Fig. 1). The finite volume analog of the liquid phase corresponds to $\lambda_n$ states with $Re(\nu_{n>0}(T)) \to \infty$.

[1] K. A. Bugaev *et al., Phys. Rev.* **C62**, 044320 (2000).
[2] K. A. Bugaev *et al., Phys. Lett.* **B 498**, 144 (2001).
[3] K. A. Bugaev, arXiv:nucl-th/0406033 (2004).